Journal of Engineering Mathematics 21: 17–22 (1987)

A note on the wall-jet problem II

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Received 29 May 1986; accepted 31 August 1986

Abstract

The wall-jet problem described in a previous paper in which the effects of suction through the wall and the wall moving were allowed so as to still satisfy the momentum condition for the wall jet is considered further. It is shown that, as well as the upper branch of solutions which were obtained previously, there is a lower branch of solutions, all of which have a region of reversed flow next to the wall.

1. Introduction

In a recent paper, [1], the authors considered a generalisation of the wall-jet problem originally studied by Glauert [2,3], in which both blowing and suction through the wall and the wall moving were allowed. It was assumed that each of these effects had the appropriate power-law variation in x (x is the non-dimensional distance along the wall) for the similarity form, obtained in [2], to be preserved. This required a transpiration velocity $v_\omega(x) = -(\alpha/4)x^{-3/4}$ and a wall velocity $U_\omega(x) = \beta x^{-1/2}$, where α and β are constants. Then, with the stream function $\psi = x^{1/4}f(\eta)$, $\eta = y/x^{3/4}$ (y is the non-dimensional co-ordinate normal to the wall), the resulting ordinary differential equation is

$$f''' + \tfrac{1}{4}ff'' + \tfrac{1}{2}f'^2 = 0 \tag{1}$$

with boundary conditions

$$f(0) = \alpha \tag{2a}$$

$$f'(0) = \beta \tag{2b}$$

$$f' \to 0 \quad \text{as } \eta \to \infty \tag{2c}$$

together with the momentum condition

$$\int_0^\infty (f-\alpha)f'^2 \, \mathrm{d}\eta = 1 \tag{2d}$$

(primes denote differentiation with respect to η). Condition (2d) arises since the form of the similarity solution is such as to make the quantity

$$\int_0^\infty \frac{\partial \psi}{\partial y}\left(\int_y^\infty \left(\frac{\partial \psi}{\partial \bar{y}}\right)^2 \mathrm{d}\bar{y}\right) \mathrm{d}y = \text{constant},$$

with the value of this constant being fixed at the leading edge of the jet.

In [1] it was shown that no solutions of equation (1) satisfying boundary conditions (2) exist when:

(i) $\alpha \neq 0, \quad \beta = 0,$
(ii) $\beta \neq 0, \quad \alpha = 0,$
(iii) $\alpha < 0, \quad$ any β,

while for certain values of α and β a solution was possible. These solutions were obtained by solving the first-order equation

$$f' = \tfrac{1}{6}\left(\sigma^3 f^{1/2} - f^2\right) \equiv G(f;\ \sigma) \tag{3}$$

where $\sigma^2 = f(\infty)$ with σ being determined in terms of α through (2d), and β determined in terms of α and σ by (2b).

In [1] the positive sign for the square root in (3) was taken throughout, giving just one possible solution for each $\alpha > 0$. However, allowing the negative sign also for the square root enables a second branch of solutions to equation (1), satisfying (2), to be constructed, again for each $\alpha > 0$. These new solutions all have a region of reversed flow next to the wall (i.e. they give rise to negative values for β). Both these branches of solution bifurcate out of Glauert's original solution at $\alpha = \beta = 0$ in the $(\alpha,\ \beta)$ plane with a square-root singularity.

2. Existence of solutions

An examination of equation (3) shows that $G(f;\ \sigma)$ is defined only in $f \geqslant 0$ where it has two branches, an upper branch in which the positive sign is taken for the square root and a lower branch in which the negative sign is taken. Also $G(f;\ \sigma)$ has zeroes at $f = 0$ and $f = \sigma^2$ and a sketch of $G(f;\ \sigma)$ is shown in Fig. 1. Due to the symmetry of $G(f;\ \sigma)$ we need only consider $\sigma > 0$ (there is no solution of (3) satisfying (2) for $\sigma = 0$), and from [1] we have to restrict attention to the case $\alpha \geqslant 0$.

Using Fig. 1 we can see that, for a given value of α, there are exactly two solutions of equation (3), one, which we will denote by $f_u\ (\eta;\ \alpha,\ \sigma)$, starting on the upper branch at $f = \alpha$ and proceeding directly to $f = \sigma^2$. The other solution, which we will denote by $f_L\ (\eta;\ \alpha,\ \sigma)$, starts on the lower branch again at $f = \alpha$, proceeds on this lower branch to $f = 0$, then follows the upper branch from $f = 0$ to $f = \sigma^2$. Using condition (2b) in (3) we obtain for the upper-branch solutions

$$\beta_u = f_u'(0;\ \alpha,\ \sigma) = \tfrac{1}{6}\left(\sqrt{\alpha}\,\sigma^3 - \alpha^2\right) \tag{4}$$

and for the lower-branch solutions

$$\beta_L = f_L'(0;\ \alpha,\ \sigma) = -\tfrac{1}{6}\left(\sqrt{\alpha}\,\sigma^3 + \alpha^2\right). \tag{5}$$

The relation between σ and α is determined by condition (2d). Consider first the upper-branch solutions. From [1] we have that (2d) gives

$$\sigma^8 - \tfrac{20}{9}\alpha\sigma^6 + \tfrac{16}{9}\alpha^{5/2}\sigma^3 - \tfrac{5}{9}\alpha^4 - 40 = 0, \tag{6}$$

which can be written as

$$(\mu - 1)^3 g(\mu) - 40\alpha^{-4} = 0 \tag{7}$$

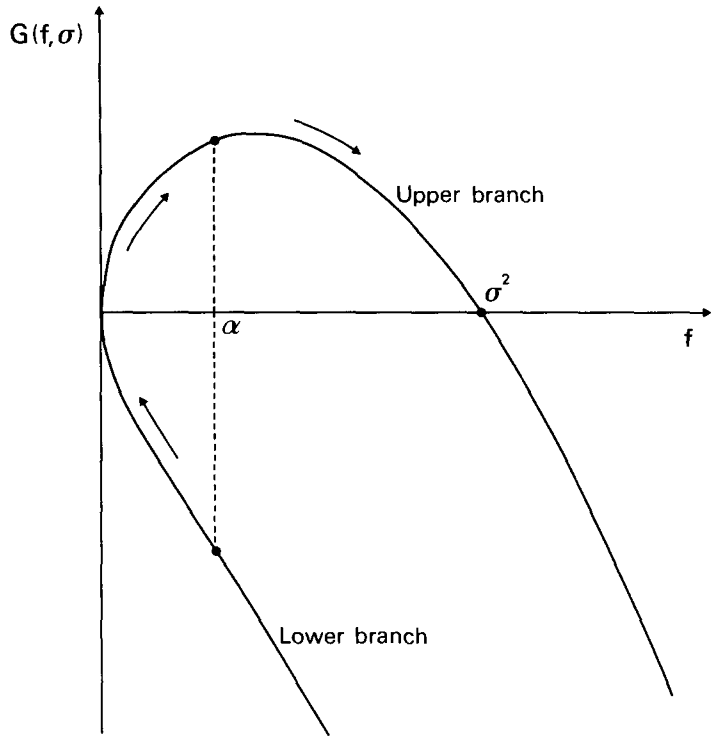

Fig. 1. A sketch of $G(f;\ \sigma)$ as defined by (3) showing the upper-branch and lower-branch solutions from $f=\alpha$.

where

$$\sigma=\mu\sqrt{\alpha} \quad \text{and } g(\mu)=\mu^5+3\mu^4+\tfrac{34}{9}\mu^3+\tfrac{10}{3}\mu^2+\tfrac{5}{3}\mu+\tfrac{5}{9}.$$

It is easy to show that $(\mu-1)^3\ g(\mu)$ is a monotone increasing function of μ in $\mu \geqslant 0$ and has just one zero at $\mu=1$, and so is negative for $0 \leqslant \mu < 1$ and positive for $\mu > 1$. Hence for a given value of $\alpha > 0$, equation (7) has just one solution, $\mu_u(\alpha)$ say, with $\mu_u(\alpha) > 1$. The corresponding value of σ, $\sigma_u(\alpha)$ say, then satisfies the condition $\sigma_u(\alpha)^2 > \alpha$ for all $\alpha > 0$ (as is borne out by the calculated values given in [1]). Also it is easy to deduce from the above that $\mu_u(\alpha)$ is a monotone decreasing function of α with $\mu_u(\alpha) \sim (40)^{1/8}\alpha^{-1/2}$ for small α and $\mu_u(\alpha) \sim 1 + 3^{1/3}\alpha^{-4/3}$ for α large, so that

$$\sigma_u(\alpha) \sim \begin{cases} \sqrt{\alpha}\,(1+3^{1/3}\alpha^{-4/3}+\dots), & \alpha \text{ large} \\ (40)^{1/8}(1+\mathrm{O}(\alpha)), & \alpha \text{ small} \end{cases} \tag{8}$$

with, from (5),

$$\beta_u(\alpha) \sim \begin{cases} \tfrac{1}{2}3^{1/3}\alpha^{2/3}+\dots, & \alpha \text{ large} \\ \tfrac{1}{6}(40)^{1/8}\sqrt{\alpha}+\dots, & \alpha \text{ small.} \end{cases} \tag{9}$$

Furthermore, since $\sigma_u^2 > \alpha$, we have that $\beta_u(\alpha) > 0$ for all $\alpha > 0$.

Consider next the lower-branch solutions, $f_L(\eta;\ \alpha,\ \sigma)$. The equation for determining σ in terms of α becomes, from (2d),

$$(\mu-1)^3\bar{g}(\mu)-40\alpha^{-4}=0 \tag{10}$$

where $\bar{g}(\mu)=g(\mu)-(64/9)\mu^3$, with $\sigma=\mu\sqrt{\alpha}$ and $g(\mu)$ as defined above.

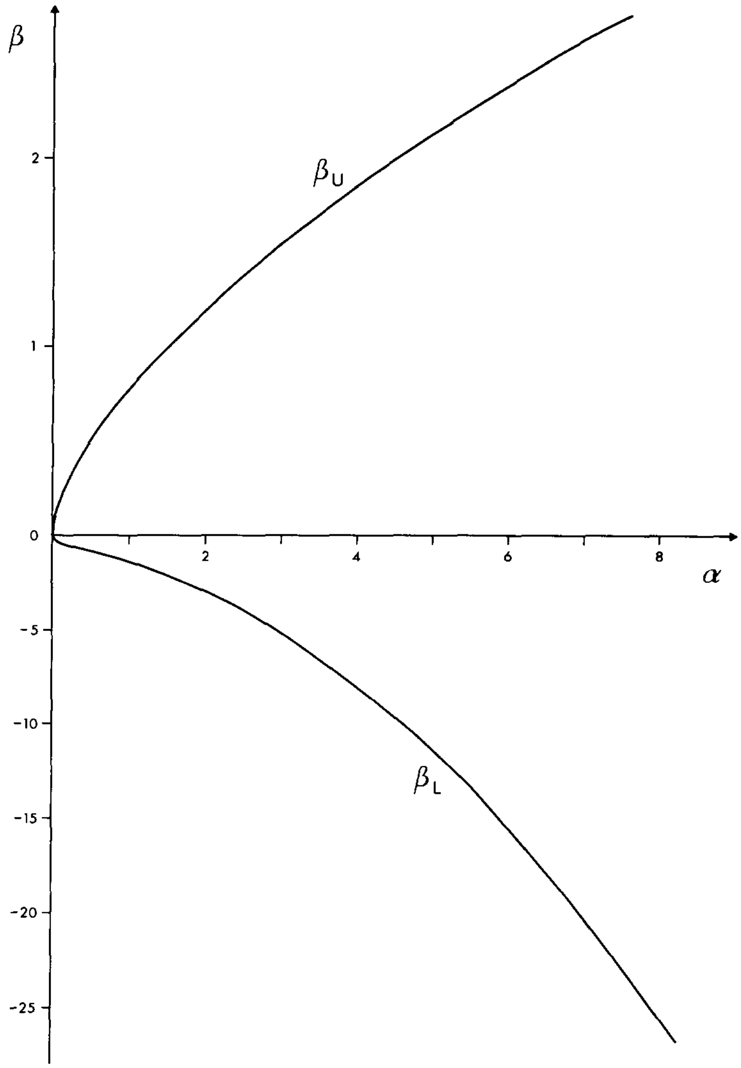

Fig. 2. Graphs of $\beta_u(\alpha)$ and $\beta_L(\alpha)$ against α (note the different scales for β_u and β_L).

Again we can deduce that $(\mu-1)^3\bar{g}(\mu)$ is a monotone increasing function of μ for $\mu \geqslant 0$ and has a single zero at $\mu = 1$. Therefore equation (10) is invertible to give μ as a single-valued, positive and monotone decreasing function of α, $\mu = \mu_L(\alpha)$ say. So for these lower-branch solutions $\sigma = \sigma_L(\alpha)$ is a positive, single-valued function of α. The behaviour of $\sigma_L(\alpha)$ for both α small and as $\alpha \to \infty$ is the same as for $\sigma_u(\alpha)$, given by (8), while the corresponding forms for $\beta_L(\alpha)$ are, using (5)

$$\beta_L(\alpha) \sim \begin{cases} -\frac{1}{3}\alpha^2 + \ldots, & \alpha \text{ large} \\ -\frac{1}{6}(40)^{3/8}\sqrt{\alpha} + \ldots, & \alpha \text{ small.} \end{cases} \tag{11}$$

Graphs of $\beta_u(\alpha)$ and $\beta_L(\alpha)$ computed numerically from solutions of equations (7) and (10) respectively are shown in Fig. 2, where, in line with (9) and (11), we can see that both branches bifurcate out of Glauert's solution at $\alpha = \beta = 0$ with a square-root singularity.

We have shown that for each $\alpha > 0$ there is just one upper-branch solution $f_u(\eta)$, and using (3) we can show that f_u is a monotone increasing function of η. Also, for each $\alpha > 0$ there is just one lower-branch solution $f_L(\eta)$. Moreover f_L decreases monotonically for η in the range $0 \leqslant \eta < \eta_0(\alpha)$ where

$$\eta_0(\alpha) = \int_0^\alpha \frac{6\,\mathrm{d}u}{\sigma^3 u^{1/2} + u^2}, \tag{12}$$

so that

$$\eta_0(\alpha) = \frac{2}{\sigma^2}\left[\log\left(\frac{(\alpha+\sigma)^2}{\alpha^2 - \alpha\sigma + \alpha^2}\right) + 2\sqrt{3}\left(\tan^{-1}\left(\frac{2\alpha - \sigma}{\sqrt{3}\,\sigma}\right) - \frac{\pi}{6}\right)\right].$$

At $\eta = \eta_0(\alpha)$ both f_L and f_L' are zero, thereafter f_L increases monotonically with η.

3. Numerical solutions

Graphs of the upper-branch solutions for various α have been given in [1] and need not be repeated here. The lower-branch solutions were obtained by integrating equation (1) numerically by a Runge-Kutta method starting at $\eta = 0$. To do this for a given value of $\alpha = f(0)$, we need to know both $f'(0)$ and $f''(0)$. Now $f'(0) = \beta_L(\alpha)$, where β_L is given by (5) with $\sigma_L(\alpha)$ as calculated by solving equation (10). (This can be done easily by Newton iteration or by prescribing a value for σ and evaluating the corresponding value for α from (10)). To calculate $f''(0)$ we differentiate (3) once with respect to η to get

$$f''(0) = -\left(\frac{\sigma_L^3 + 4\alpha\sqrt{\alpha}}{12\sqrt{\alpha}}\right)\beta_L, \tag{13}$$

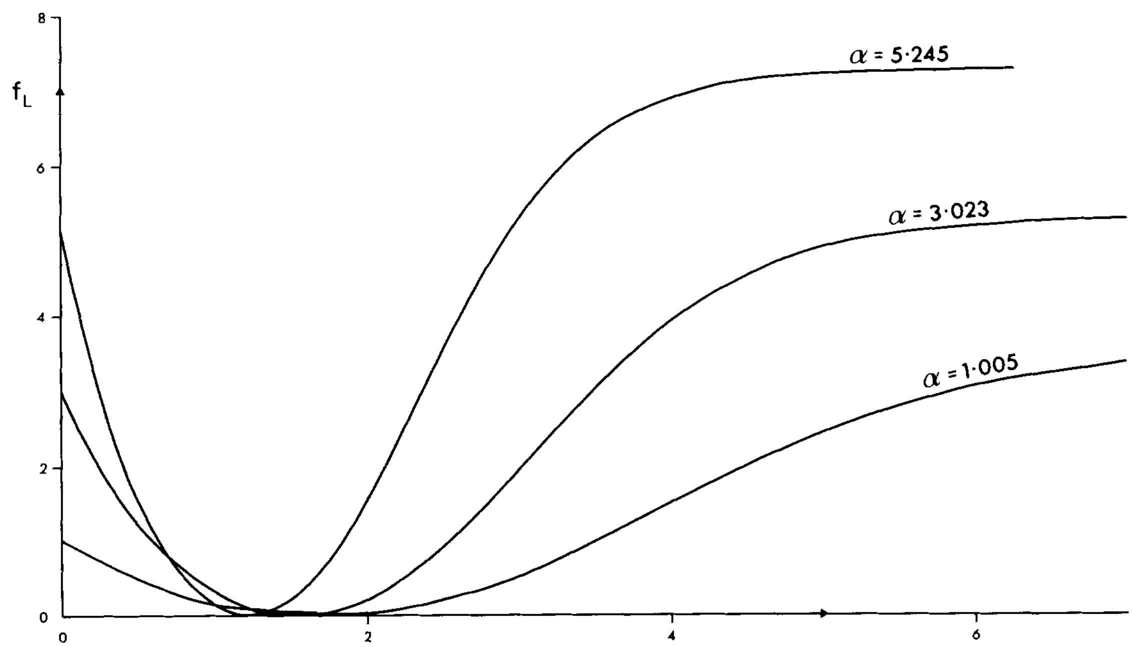

Fig. 3. Graphs of $f_L(\eta)$ against η for various α.

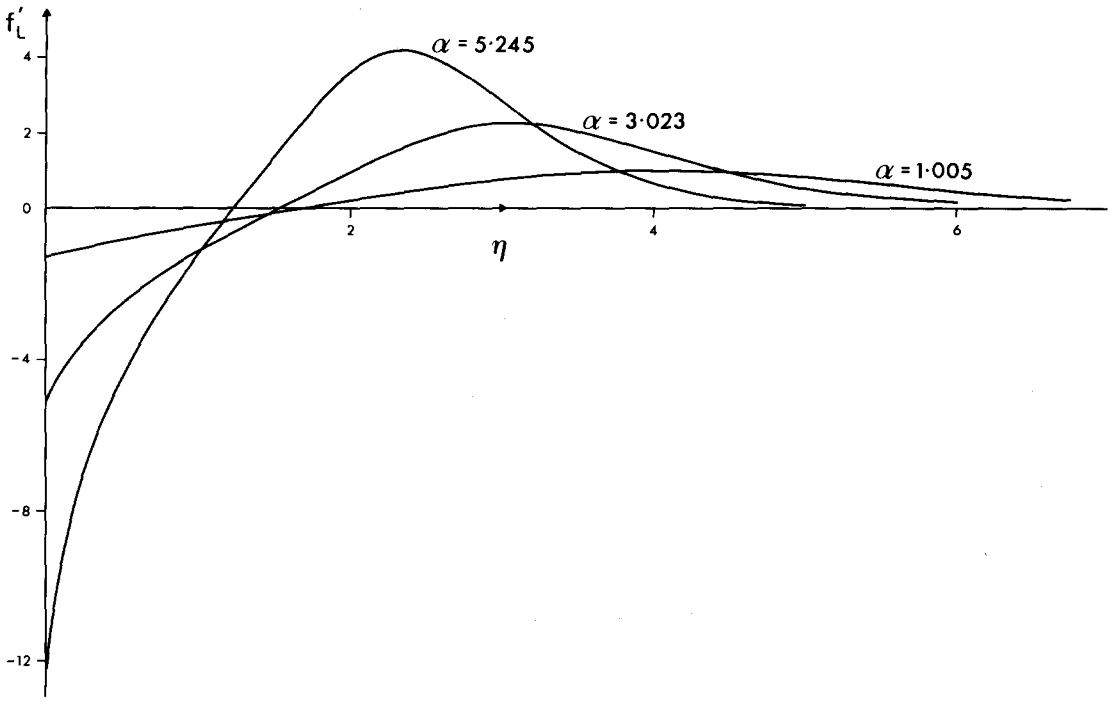

Fig. 4. Graphs of $f_L'(\eta)$ against η for various α.

which can be calculated directly. Then with $f(0)$, $f'(0)$ and $f''(0)$ known, equation (1) can be integrated numerically by a direct marching method. Graphs of f_L and f_L' for various α are shown in Figs. 3 and 4 respectively. These graphs clearly show the region of reversed flow next to the wall.

References

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